A relativistic quark model for mesons

1974 J. Phys. A: Math. Nucl. Gen. 7 1681

(http://iopscience.iop.org/0301-0015/7/14/001)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 14, 1974. Printed in Great Britain. © 1974

A relativistic quark model for mesons

P J Walters†, A M Thomson and F D Gault
Physics Department, University of Durham, South Road, Durham, UK

Received 12 March 1974, in final form 30 April 1974

Abstract. A relativistic quark model, which is closely related to the non-relativistic harmonic oscillator model, is presented for mesons with spin-$\frac{1}{2}$ quarks. The conventional quark model spectrum is reproduced, with particles lying on straight line trajectories, although for $PC = 1$ states large masses are predicted. The relativistic approach used throughout gives rise naturally to spin–orbit terms. These terms are analogous to those which arise in the Melosh approach. The comparisons of the model predictions with data is encouraging and the results include meson decay widths for emission of a pseudoscalar meson or a photon.

1. Introduction

The ability of the symmetric, non-relativistic harmonic oscillator model to classify mesons and baryons and to describe some features of their interactions is well established (Gell-Mann 1964, Greenberg 1964, Dalitz 1966, 1967, Faiman and Hendry 1968, 1969, Copley *et al* 1969, Katyal and Mitra 1970, Choudhury and Mitra 1970) and there have been several attempts to derive a relativistic generalization of the model (Feynman, Kislinger and Ravndal (FKR) 1971, Rosner 1972, Bohm, Joos and Krammer (BJK) 1973). We will also produce a relativistic generalization but we confine ourselves to mesons.

In the meson case one has a bound-state problem for a quark and antiquark and field theory suggests the Bethe–Salpeter (BS) equation as a starting point. Unfortunately, this equation is difficult to solve exactly, even for simple interactions (Sundaresan and Watson 1970) and so some approximation scheme is required if we are to abstract the systematics rather than get involved in numerical solutions. The assumption that the quark mass is large provides a basis for an approximate solution.

BJK demonstrate one approach where they solve the zero mass problem exactly for the harmonic oscillator interaction, and then find the finite mass solutions by a perturbation expansion in $M_{\text{hadron}}/M_{\text{quark}}$. We, on the other hand take the infinite quark mass limit of the BS equation from the outset (Susskind 1968). We do this in order to generate a relativistic expression similar in form to that of the non-relativistic quark model. The resulting linear equation is more tractable and we simplify and define the problem further by demanding that the squared equation has harmonic oscillator form. This ensures that the equation is easy to solve, but makes the interactions less realistic. Once we have arrived at this linear equation we regard it as the model and view the quarks as inseparable from the hadron. Of course, in this view the quark mass can have little meaning.

† Present Address: Department of Natural Philosophy, The University, Glasgow, UK.

We are able to calculate the quark mass that enters our model from pion decay and it turns out to be the same order as the pseudoscalar meson mass, just as in the non-relativistic model. Although this would appear to invalidate our approximation scheme, we have taken a more abstract view of quarks as forming a simple representation of the internal degrees of freedom of the hadron.

From this point of view a linear equation could be more appropriate than a BS equation. The motivation for this assertion is found in dual models (Susskind 1968). Indeed, our original aim was to build a model which was like the lowest order term in a dual quark model. For our purposes the Ramond model (Ramond 1971) is most suitable as it describes fermions and enables us to have spin-$\frac{1}{2}$ quarks. In this model, one has a linear equation in which the basic interaction is also linear in a set of orbital oscillator variables which are contracted with a set of spin excitation operators. Our model has just these features, but we have only one orbital degree of freedom whereas the dual models have an infinite number.

The model of FKR may also be considered as a first step in a dual quark model, but there the analogy is with the Fubini–Veneziano model rather than the Ramond model as spin is not incorporated dynamically. As the FKR model is most closely related to our model, we compare and contrast the two approaches throughout the paper.

Spin–orbit terms appear naturally in our formalism and affect our results in a comparable way with that in which the Melosh transformation (Melosh 1973) between current and constituent quarks modified the non-relativistic results. Such terms also occur in FKR but in the currents, rather than in the states, as in our case, and then only in the axial current.

When dealing with the 0^- nonet we will assume no η, η' mixing and for the 1^- nonet we take ω, ϕ to be ideally mixed.

2. The model

Consider two quarks of momenta p_1 and p_2 bound by a covariant potential V. The BS equation is

$$(\not{p}_1 - m_q)\phi(\not{p}_2 - m_q) = V\phi \tag{2.1}$$

and if the quark mass m_q is large the equation can be approximated by (Susskind 1968)

$$(\not{p}_1 + \not{p}_2)\phi = \left(m_q - \frac{V}{m_q}\right)\phi = (m_0 - U)\phi \tag{2.2}$$

where m_0 is the effective quark mass. Here we consider only the quark–quark equation. The quark–antiquark result is derived using the charge conjugation matrix. Substituting for the centre of mass and relative momentum variables P, q given by $p_1 = \frac{1}{2}P + q$, $p_2 = \frac{1}{2}P - q$, the equation becomes

$$[\tfrac{1}{2}P_\mu(\gamma_1^\mu + \gamma_2^\mu) + q_\mu(\gamma_1^\mu - \gamma_2^\mu) - m_0 + U]\phi = 0. \tag{2.3}$$

Our four-vector conventions are those of Bjorken and Drell (1964).

We choose the form of the interaction in the same way as FKR. As one can derive their model from a BS equation for scalar quarks

$$(p_1^2 - m_q^2)\phi(p_2^2 - m_q^2) = V\phi \tag{2.4}$$

giving as $m_q \to \infty$

$$\left[p_1^2+p_2^2-m_q^2+\frac{V}{m_q^2}\right]\phi = 0 \tag{2.5}$$

the method is strictly analogous. For harmonic oscillator interaction this becomes

$$[p_1^2+p_2^2+2\omega^2x^2+c]\phi = 0$$

$$[\tfrac{1}{2}P^2+2(q^2+\omega^2x^2)+c]\phi = 0 \tag{2.6}$$

where $x^\mu = -\mathrm{i}(\partial/\partial q_\mu) = x_1^\mu - x_2^\mu$, is the relative separation of the quarks and 'c' is a constant. Introducing creation and annihilation operators

$$a_\mu = \frac{1}{\sqrt{(2\omega)}}[q_\mu - \mathrm{i}\omega x_\mu],$$

$$a_\mu^\dagger = \frac{1}{\sqrt{(2\omega)}}[q_\mu + \mathrm{i}\omega x_\mu], \tag{2.7}$$

we obtain

$$K = \tfrac{1}{2}[P^2-M^2] = \tfrac{1}{2}[P^2+\Omega a_\mu^\dagger a^\mu + 2c] \tag{2.8}$$

where $\Omega = 8\omega$ and $P^2 = M^2$ is the meson mass squared.

To obtain the interaction for spinor quarks we linearize in the spin space $\frac{1}{2}\otimes\frac{1}{2}$ to get

$$\tfrac{1}{2}P_\mu(\gamma_1^\mu+\gamma_2^\mu) \sim M \sim \tfrac{1}{2}\sqrt{\Omega}(\alpha_+ a^\dagger + \alpha_- a^\dagger). \tag{2.9}$$

To ensure that the squared equation has oscillator form, we must eliminate terms quadratic in a^μ and $a^{\mu\dagger}$ by the conditions

$$\alpha_+^\mu \alpha_+^\nu = \alpha_-^\mu \alpha_-^\nu \equiv 0. \tag{2.10}$$

Equation (2.3) determines the way in which the internal momenta enter equation (2.9).

These constraints suggest the form

$$\alpha_\pm^\mu = \tfrac{1}{2}(I+P_\pm)(\gamma_1^\mu - \gamma_2^\mu) \tag{2.11}$$

where Lorentz and parity invariance indicate how $P_\pm$ can be expanded in terms of Fermi bilinear covariants. Using equation (2.11) one can deduce that

$$\alpha_\pm^\mu = \tfrac{1}{2}(I \pm \gamma_5^1\gamma_5^2)(\gamma_1^\mu - \gamma_2^\mu) \tag{2.12}$$

and the square root equation becomes

$$[\tfrac{1}{2}P_\mu(\gamma_1^\mu+\gamma_2^\mu) - m_0 + \tfrac{1}{2}\sqrt{\Omega}(\alpha_+ a + \alpha_- a^\dagger)]\phi = 0. \tag{2.13}$$

Explicitly

$$\tfrac{1}{2}\sqrt{\Omega}(\alpha_+ a + \alpha_- a^\dagger) = (\gamma_1^\mu - \gamma_2^\mu)q_\mu - \mathrm{i}\omega\gamma_5^1\gamma_5^2(\gamma_1^\mu - \gamma_2^\mu)x_\mu \tag{2.14}$$

we deduce that

$$U(x_1, x_2) = -\mathrm{i}\omega\gamma_5^1\gamma_5^2(\gamma_1^\mu - \gamma_2^\mu)(x_1 - x_2)_\mu. \tag{2.15}$$

This form of the interaction, linear in position variables, which, like the momenta, are contracted with Dirac matrices can be seen to arise from the position momentum symmetry characteristic of the harmonic oscillator. Equation (2.14) is clearly invariant under the interchange $q_\mu \leftrightarrow -\mathrm{i}\omega\gamma_5^1\gamma_5^2 x_\mu$.

We note that because the interaction $U(x_1, x_2)$ does not contain a scalar potential we are unable to cancel the large quark mass, m_q, on which we have based our derivation, to get a small effective quark mass m_0. The apparent inconsistency which arises from m_0 being in fact m_q we shall ignore in order to take advantage of the $m_q \to \infty$ ansatz. The obvious advantage is the elimination of quark propagator effects which establishes a close relation between our relativistic results and non-relativistic results. This philosophy is entirely consistent with that of FKR (scalar quarks against our spinor quarks) and our desire to establish a calculational prescription from which we can abstract systematics.

A further advantage of the infinite quark mass picture is that we can regard the internal motion of the constituent quarks as non-relativistic (although we are then obliged to neglect the relative energy in the centre of mass). This allows us to replace the four-dimensional operators, a_μ, by

$$\eta_\mu = a_\mu - P_\mu \frac{P \cdot a}{m^2}.$$

As $\eta_\mu = (0, \boldsymbol{a})$ in the rest frame, the close relationship with the non-relativistic model is clear. As well, the negative norm time-like states are eliminated. These states in the FKR model are decoupled from physical states by a gauge condition and this has the effect of violating unitarity. As a consequence, the FKR matrix elements are too large and they are obliged to reduce them by using 'an adjustment factor'.

3. Properties of the wavefunctions

The solutions to equation (2.13) can be built up in a Fock space with a vacuum $|0_p\rangle$ defined by

$$\eta_\nu |0_p\rangle = 0. \tag{3.1}$$

This is just the ordinary non-relativistic oscillator ground state in the rest frame. A general solution of equation (2.13) can then be written as

$$|\psi_p\rangle = \psi_p(\eta, \eta^\dagger)|0_p\rangle \tag{3.2}$$

where ψ_p is a 4×4 matrix. In calculations, quark operators act on ψ_p from the right, while antiquark operators act from the left. The invariance properties of the solutions under charge conjugation, Lorentz and parity transformations are defined in the same way as for the BS wavefunctions. In particular, charge conjugation is defined by

$$C\psi_p^{\mathrm{T}}(-\eta, -\eta^\dagger)C^{-1}|0_p\rangle = \eta_c \psi_p(\eta, \eta^\dagger)|0_p\rangle. \tag{3.3}$$

Because the wavefunction does not depend on the relative energy (a consequence of the infinite quark mass approach) we can not use the invariant measure d^4q. This requires redefinition of the scalar product which we do in a way motivated by FKR. That is, we replace $|0_p\rangle$ by $|0\rangle$ the four-dimensional vacuum state, which also satisfies (3.2). The spin part of the scalar product is defined in the usual way, so we have

$$\langle \phi_{p_2}|\psi_{p_1}\rangle = \langle 0|\mathrm{Tr}\, \bar{\phi}_{p_2}(\eta_2^\dagger, \eta_2)\psi_{p_1}(\eta_1, \eta_1^\dagger)|0\rangle$$

where

$$\bar{\phi}_{p_2}(\eta_2^\dagger, \eta_2) = \gamma_0 \phi_{p_2}^\dagger(\eta_2, \eta_2^\dagger)\gamma_0. \tag{3.4}$$

We normalize the wavefunctions by using the matrix elements of the quark current at zero momentum transfer. These define the quark charge, e_q, and give

$$\left\langle \psi_p \left| \frac{j_\mu^q(0)}{e_q} \right| \psi_p \right\rangle = -\langle 0|\mathrm{Tr}\,\bar{\psi}_p(\eta^\dagger, \eta)\gamma_\mu \psi_p(\eta, \eta^\dagger)|0\rangle \equiv 2P_\mu. \tag{3.5}$$

We compare this with the corresponding BS normalization (figure 1).

$$\left\langle \psi_p \left| \frac{j_\mu^q(0)}{e_q} \right| \psi_p \right\rangle = -\mathrm{Tr}\,\langle \bar{\psi}_p \gamma_\mu \psi_p (\tfrac{1}{2}\not{P} + \not{k} + m_q)\rangle \tag{3.6}$$

which for large m_q becomes $\sim -m_q \mathrm{Tr}\langle \bar{\psi}_p \gamma_\mu \psi_p \rangle$ and we note that our normalization in (3.5) is consistent with the general approach.

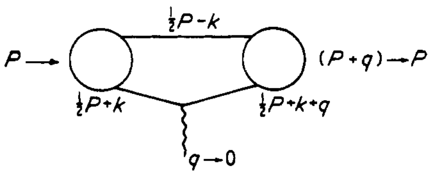

Figure 1. Current normalization in the Bethe–Salpeter equation.

As is obvious from the above, the minimal vector current for the quarks is

$$V_\mu^q = -e_q \gamma_\mu e^{iq\cdot x} = -e_q \gamma_\mu F \exp\left(\frac{q\cdot a^\dagger}{\sqrt{\Omega}}\right) \exp\left(\frac{-q\cdot a}{\sqrt{\Omega}}\right) \tag{3.7}$$

with $F = \exp(-q^2/2\Omega)$ and the centre of mass phase factor neglected. The axial current is

$$A_\mu^q = -\lambda_q \gamma_5 \gamma_\mu F \exp\left(\frac{q\cdot a^\dagger}{\sqrt{\Omega}}\right) \exp\left(\frac{-q\cdot a}{\sqrt{\Omega}}\right). \tag{3.8}$$

Neglecting the unitary spin factor the antiquark operators are given by

$$O_\mu^{\bar{q}} = FC\Gamma_\mu^{\mathrm{T}} \exp\left(\frac{-q\cdot a^\dagger}{\sqrt{\Omega}}\right) \exp\left(\frac{q\cdot a}{\sqrt{\Omega}}\right) C^{-1} \tag{3.9}$$

where Γ_μ is either γ_μ or $\gamma_5\gamma_\mu$. The corresponding antiquark matrix elements are given by

$$\langle \phi_{p_2}|0^{\bar{q}}|\psi_{p_1}\rangle = F\left\langle 0 \left| \mathrm{Tr}\,\bar{\phi}(\eta_2^\dagger, \eta_2) \exp\left(\frac{-q\cdot a^\dagger}{\sqrt{\Omega}}\right) \exp\left(\frac{q\cdot a}{\sqrt{\Omega}}\right) \psi_{p_1}(\eta_1, \eta_1^\dagger) C\Gamma^{\mathrm{T}} C^{-1} \right| 0 \right\rangle.$$

4. The solutions

The solutions for the model were found by the direct method of writing equation (2.13) as a set of coupled equations at the Pauli spinor level in the rest frame, and algebraically reducing them to a single equation. The restrictions of charge conjugation and parity were applied from the outset to divide the solutions into two classes characterized by the value of PC. The classes are $PC = -1$ which corresponds in the non-relativistic quark model to spin zero, and $PC = +1$ which corresponds to spin one. This relation between PC and quark spin is maintained in the relativistic wavefunctions, but only in general by large components. With some manipulation one finds that the large components are eigenstates of the number operator and this is used to construct the full solution. This solution, re-written in covariant form, can then be verified by substituting it back into equation (2.13). Details are given in the appendix.

4.1. $PC = -1$

We reproduce the non-relativistic result that the $C = (-1)^{J+1}$, $P = (-1)^J$ mesons are forbidden and this follows directly from the requirement that the interaction be three dimensional. In general however, the interaction can be four dimensional, in which case this result becomes strongly dependent on the spin-space structure of the interaction (Bohm *et al* 1973).

The solution for a $PC = -1$ state of mass m and spin quantum numbers (J, J_Z) is

$$\frac{1}{2\sqrt{m_0}}\gamma_5\left[\left(m - m_0\frac{\not{P}}{m}\right) + \sqrt{\Omega}\frac{\not{P}}{m}\not{\eta}\right]|N_J, J, J_Z\rangle \tag{4.1}$$

for $C = (-1)^J$, $P = (-1)^{J+1}$. $|N_J, J, J_Z\rangle$ is an eigenstate of the harmonic oscillator number operator, $-\eta_\mu^\dagger\eta^\mu$, with eigenvalue N, and J, J_Z are orbital angular momentum quantum numbers. We note that in the limit $\Omega \to 0$ the solution reduces to that for a free (mass m_0) quark and antiquark with relative momentum zero and consequently there is no admixture of undesirable negative energy states. We also note that the ground state pseudoscalar solution,

$$\frac{1}{2\sqrt{m_0}}\gamma_5\left(m - m_0\frac{\not{P}}{m}\right)|0\rangle \tag{4.2}$$

is simply a boosted non-relativistic solution.

The particles lie on straight Regge trajectories:

$$m^2 = m_0^2 + \Omega N, \qquad N = 0, 1, 2, \ldots . \tag{4.3}$$

If we choose the slope of the trajectory to be $\Omega^{-1} = 1$ and take for m_0^2 the average of the square of the pseudoscalar meson masses, $m_0^2 = 0{\cdot}25\ \mathrm{GeV}^2$ we reproduce reasonably this section of the mass spectrum (figure 2).

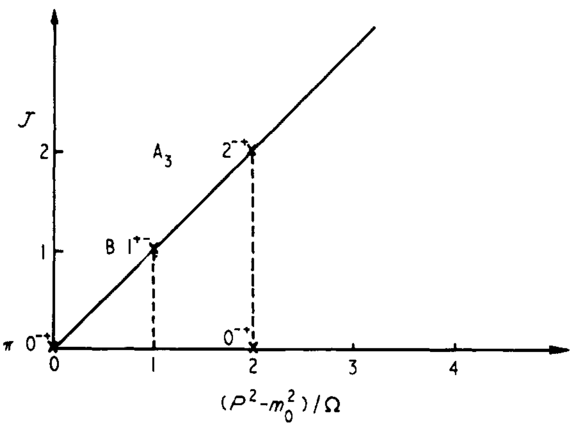

Figure 2. Predicted mass spectrum for $PC = -1$ mesons.

The presence of terms containing the inverse of the pseudoscalar mass gives rise to large symmetry breaking and the Van Royen–Weisskopf paradox (Van Royen and Weisskopf 1967a, b). As this results in unreasonable predictions and not only for pseudoscalar meson decay matrix elements, we avoid it by using predicted rather than physical masses in equation (4.1). Then the 0^- solution (4.2) has the same spin structure as that postulated by Gudeus (1969).

In the BS model of BJK the 0^- solutions have a similar form but $\not{P}/m$ is replaced by $\not{P}/m_{\text{quark}}$. This has the advantage of resolving the Van Royen–Weisskopf paradox in a natural way (Llewellyn-Smith 1969).

Finally, it is interesting to observe that in fixing the slope parameter, we have fixed the size of the hadron (Le Yaouanc *et al* 1972). The space part of the ground state wavefunction is

$$\langle \boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2 | 0_{p=m} \rangle \sim \exp\left(\frac{-\omega r^2}{2}\right) \sim \exp\left(\frac{-r^2}{2R^2}\right) \tag{4.4}$$

where $R^2 = 1/\omega = 8/\Omega \sim 8$ natural units. R is essentially the average separation of the quarks and determines the size of the meson. The meson cross section is $\pi R^2 \sim 24$ natural units ~ 10 mb, which is not unreasonable.

4.2. $PC = +1$

In solving for the $PC = +1$ we have the added difficulty that the solutions are not, in general, unique. However, for special cases, including the leading trajectory, the states are unambiguous. These cases occur when the spin is equal to the orbital angular momentum in the large components.

The problem of the ambiguity arises because the eigenvalue equation at the Pauli spinor level demands an eigenstate of the number operator. As a direct consequence the quark spin and the orbital angular momentum are decoupled and for a given meson spin two possible values of orbital angular momentum are allowed. This degeneracy has the compensation that there is no spin–orbit splitting. Also, as the first such case occurs for a spin-one meson with $N = 2$ the effect is irrelevant for practical calculations. The criterion we impose to get a general solution is to require the rest frame wavefunction to have no spin-zero component, in accord with the unambiguous solutions.

The unnormalized $PC = +1$ states are then given by

$$\frac{1}{2\sqrt{m_0}}\left[\left(m + m_0\frac{\not{P}}{m}\right)\left(\not{\epsilon} + \Omega\frac{\not{\eta}^{\dagger}\epsilon\,.\,\eta}{m^2 - m_0^2}\right) - \mathrm{i}\sqrt{\Omega}\frac{\gamma_5}{m}\mathscr{E}(P\eta^{\dagger}\epsilon\gamma)\right]|N, l, l_z\rangle \tag{4.5}$$

where $C = P = (-1)^l$, ϵ_μ is a spin-one wavefunction and $\mathscr{E}(P\eta^{\dagger}\epsilon\gamma)$ is the Levi-Civita tensor dotted into four four-vectors. The solution has not been separated into its possible spin states of $l+1$, $l-1$ and l. As in the $PC = -1$ solution we note that in the limit $\Omega \to 0$ the solution reduces to that for a free quark and antiquark (mass m_0) with relative momentum zero and we are able to conclude that there is no negative energy admixture.

The term $\not{\eta}^{\dagger}\epsilon\,.\,\eta/(m^2 - m_0^2)$, which vanishes in the case $J = l$, is responsible for adding the extra orbital angular momentum in the large components for the cases, $J = l \pm 1$, where the solution is not unique.

In order to understand the significance of the other terms in equation (4.5) it is easiest to consider the ground state vector meson solution

$$\frac{1}{2\sqrt{m_0}}\left[\left(m + m_0\frac{\not{P}}{m}\right)\not{\epsilon} - \mathrm{i}\sqrt{\Omega}\frac{\gamma_5}{m}\mathscr{E}(P\eta^{\dagger}\epsilon\gamma)\right]|0\rangle. \tag{4.6}$$

The term involving $[m+(m_0\not{P}/m)]$ is closely related to a boosted non-relativistic wave-function. In fact, if $m_0 = m$ it is just that and the term reduces to the solution of Gudeus (1969). The vector meson solution of BJK,

$$\left(1+\frac{\not{P}}{m_{\text{quark}}}\right)\not{\epsilon}|0\rangle, \qquad \langle q|0\rangle = \exp\left(\frac{-q^2}{2\Omega}\right) \tag{4.7}$$

is also similar to this term.

In the last term of equation (4.6), the quark spin is coupled to a P-wave orbital state (in the small components) and hence corresponds to a spin–orbit interaction. This feature is absent from the solutions of BJK and Gudeus. Experiments, on the other hand, indicate the necessity of some spin–orbit coupling, even if it is not the amount predicted and consequently we regard this property of the solution as most desirable. The presence of this spin–orbit interaction does not contradict our previous assertion that there is no spin–orbit splitting.

The mass spectrum is again linear and independent of any arbitrariness of the solutions. It is

$$m^2 = m_0^2+\Omega(N+2). \tag{4.8}$$

With the parameters given previously the mass of the vector meson is predicted to be 1·58 GeV, or about twice the correct answer. Despite the fact that this result, which comes from a strong spin–spin interaction, is much too big, it is clear that spin–spin coupling is needed to make the vector mesons more massive than the pseudoscalar nonet. In calculations we will take the $PC = 1$ trajectory to be

$$m_+^2 = m_0^2+\Omega(N+0{\cdot}5). \tag{4.9}$$

We have chosen the numbers so that the average value of the vector meson mass squared is approximately correct. Also, as might be anticipated in view of the non-relativistic approximation of the internal motion, the quantum number spectrum displayed in figure 3 is identical to that given by the non-relativistic quark model.

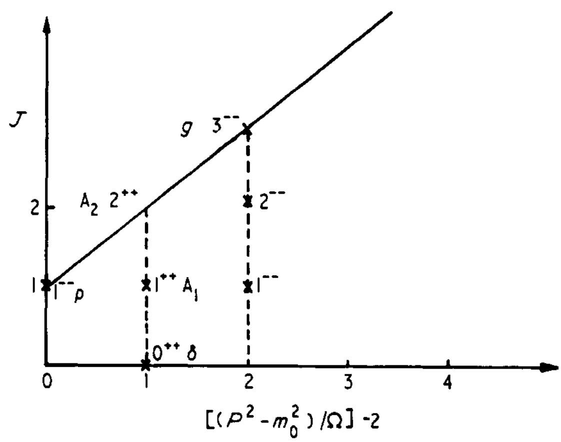

Figure 3. Predicted mass spectrum for $PC = +1$ mesons.

5. Applications

Following FKR we calculate experimental observables from current matrix elements, neglecting quark propagation effects ($m_q \to \infty$). Unlike FKR, we have a problem with our vector currents as they are not, in general, conserved. For particular cases however, they are and these cover most of the important applications. The specific cases where the electromagnetic current is conserved are the equal mass transition and the $PC = +1$ to pseudoscalar meson transition. For the rest, we add a momentum term to the coupling to ensure conservation.

5.1. Lepton decays of pseudoscalar and vector mesons

From the correspondence in § 3 between the BS and current normalization we have $\psi_{BS} = a\psi$ where $a = 1/\sqrt{m_q}$. The pseudoscalar meson decay matrix element is then given by

$$\langle Vac|A_\mu(0)|0^{-+}\text{meson}\rangle = a\text{Tr}\gamma_5\gamma_\mu\psi_p(x_\mu = 0) \equiv f_p P_\mu \tag{5.1}$$

$\psi_p(x_\mu = 0)$ is the pseudoscalar wavefunction at the origin. The result is

$$f_p = a\frac{\sqrt{m_0}}{m_p}\left(\frac{\Omega}{4\pi}\right). \tag{5.2}$$

As already mentioned, we cannot use the physical masses because of the $1/m_p$ term which will give rise to the large symmetry breaking in the Van Royen–Weisskopf paradox. We avoid this difficulty by using unbroken masses, $m_p = m_0$ which give

$$f_k = f_\pi = 0{\cdot}112a \text{ GeV}. \tag{5.3}$$

Comparing with experimental values (Particle Data Group 1973) $f_k^{exp} = 0{\cdot}105$ GeV, $f_\pi^{exp} = 0{\cdot}095$ GeV we see that 'a' is compatible with unity and a small quark mass. This is consistent with the earlier identification of m_q and m_0 but incompatible with the heavy quark approximation. However, we are not using a heavy quark approximation in our calculation of matrix elements, rather we are using it to generate relativistic expressions similar in form to those of the non-relativistic quark model. The quarks themselves are regarded as being inseparable from the hadron where the quark mass can have little meaning.

Similarly, the vector meson decay to a pair of leptons is predicted. Neglecting the unitary spin factor, we have

$$\langle Vac|V_\mu(0)|1^{-+}\text{meson}\rangle = a\text{Tr}\gamma_\mu\psi_r(x = 0) = g_v m_v \epsilon_{v\mu} \tag{5.4}$$

which gives the result

$$\frac{3}{\sqrt{2}}g_\phi = 3g_\omega = g_\rho = \frac{a}{\sqrt{(2m_0)}}\left(\frac{\Omega}{4\pi}\right) = 0{\cdot}08a \text{ GeV}. \tag{5.5}$$

The experimental result is

$$\frac{3}{\sqrt{2}}g_\phi = 0{\cdot}168 \text{ GeV}, \qquad 3g_\omega = 0{\cdot}156 \text{ GeV}, \qquad g_\rho = 0{\cdot}160 \text{ GeV}. \tag{5.6}$$

Taking $a = 1$ we get about half the experimental result which is comparable with other quark models. For example BJK predict g_v to be about twice the experimental value with the quark mass determined from pion decay, while FKR predict $g_\pi = g_k = g_\rho$ etc.

5.2. *Matrix elements of the vector current*

The form for Kπ coupling in K_{l_3} decay is calculated from the matrix element of the vector current

$$\langle \pi | V_\mu(0) | \mathrm{K} \rangle = \frac{F}{2\sqrt{2}} \left[\left(\frac{m_\pi}{m_k} + \frac{m_k}{m_\pi} \right) (P_k + P_\pi)_\mu - \left(\frac{m_k}{m_\pi} - \frac{m_\pi}{m_k} \right) (P_k - P_\pi)_\mu \right] \tag{5.7}$$

$$\equiv f_+(q)(P_k + P_\pi)_\mu + f_-(q)(P_k - P_\pi)_\mu .$$

To be consistent, we evaluate (5.7) using unbroken masses, $m_\pi = m_k = m_0$ and predict $f_-(q^2) = 0$. This does not agree with the experimental result of $f_-(0) \simeq f_+(0) \simeq 1{\cdot}0$ (Particle Data Group 1973). As this disagreement is attributable to current conservation and the use of unbroken masses it is clear that a symmetry breaking scheme must be added to our model if these results are to be well described. Had we used physical masses in (5.7) we would have achieved good predictions but violated our prescription for calculation.

We have more success in predicting the electromagnetic decays of the vector mesons. Neglecting the unitary spin factor we obtain for the matrix element of the quark current

$$T^q = -\mathrm{i} \langle \pi | \epsilon^{\mu *} J_\mu^q(0) | 1^{+-} \text{meson} \rangle$$

$$= -\frac{e m_2}{m_1 m_0} F \left[1 + \left(\frac{m_0}{m_2} \right)^2 \right] \mathscr{E}(q \epsilon_2^* p_1 \epsilon_1) \tag{5.8}$$

where the kinematics are indicated in figure 4.

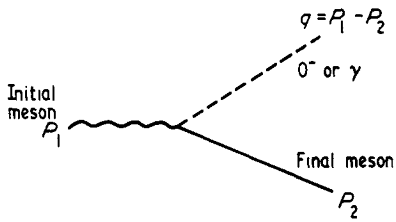

Figure 4. Kinematics of decay by emission of a photon or pseudoscalar meson.

The first term in equation (5.8) corresponds to an orbital magnetic moment, while the second is analogous to an intrinsic moment. The antiquark amplitude differs only in sign, so combining with the unitary spin factor we get the results in table 1.

Table 1. Predicted coupling constants for electromagnetic decays compared with data from Ebel *et al* (1971)

Decay	g_γ(GeV^{-1})	$g_\gamma^{\exp}$(GeV^{-1})
$\omega^0 \to \pi^0\gamma$	$-2{\cdot}2$	$-2{\cdot}89 \pm 0{\cdot}26$
$\phi^0 \to \pi^0\gamma$	0	$-0{\cdot}16 \pm 0{\cdot}02$
$\phi^0 \to \eta^0\gamma$	$-1{\cdot}2$	$-0{\cdot}82 \pm 0{\cdot}12$

The coupling constant, g_γ, is defined by

$$T = eg_\gamma \mathscr{E}(q\epsilon_2^* p_1 \epsilon_1) \tag{5.9}$$

and we have used unbroken masses in its evaluation. The quantities dotted into the Levi-Civita tensor are taken at their physical values, primarily for convenience, as any other procedure has a minor effect on the results.

5.3. Meson decays by emission of a pseudoscalar meson

Following FKR we calculate the amplitude for pseudoscalar meson emission by replacing the pseudoscalar meson interaction by the divergence of the axial vector current which is given by

$$\partial_\mu A_q^\mu = -i\lambda_q F\gamma_5 \not{q} \exp\left(\frac{q \cdot a^\dagger}{\sqrt{\Omega}}\right) \exp\left(\frac{-q \cdot a}{\sqrt{\Omega}}\right). \tag{5.10}$$

In the decays $1^+ \to 1^-0^-, 2^+ \to 1^-0^-$ the corresponding vector current is not conserved and consequently we should adjust the axial vector current to be

$$\partial_\mu A_q^\mu = -i\lambda_q F\gamma_5[g_1\not{q} + g_2 q \cdot (p_1+p_2)] \exp\left(\frac{q \cdot a^\dagger}{\sqrt{\Omega}}\right) \exp\left(\frac{-q \cdot a}{\sqrt{\Omega}}\right) \tag{5.11}$$

where g_1, g_2 are constrained to ensure electromagnetic current conservation and $g_1^2 + g_2^2 = 1$. For the actual predictions presented in table 4, we use equation (5.10).

The overall strength of the interaction is determined by one adjustable parameter f, and the decay amplitude from an initial state, i, to a final state, f, by emission of a pseudoscalar is

$$T = -if \sum_\alpha \langle \mathrm{f}|\partial_\mu A_\alpha^\mu|\mathrm{i}\rangle \tag{5.12}$$

where the summation is over quarks and antiquarks. Expressions for the matrix elements with the unitary factor λ_q removed are given in table 2.

Decay widths are calculated using the formula

$$\Gamma = \frac{R}{(2J_i+1)} \frac{\Sigma|T|^2}{2M_1^2}|\boldsymbol{q}| \tag{5.13}$$

where $\boldsymbol{q}$ is the three momentum of the decay products (in the rest frame of the initial particle i) and the summation is over the initial and final spins. The factor R is to account for the different charge modes allowed in the decay (see FKR).

In calculating the decay widths the dynamical quantities, the decay constants, are input using unbroken masses, while for the kinematic phase space, we use physical masses. For example, the decay width

$$\Gamma(1^{--} \to 0^{-+}0^{-+}) = R\frac{2}{3}\frac{g^2}{m_1^2}|\boldsymbol{q}|^3 \tag{5.14}$$

is calculated by putting physical masses into $|\boldsymbol{q}|$ and unbroken masses into (g/m_1).

As can be seen from table 3, this prescription yields good agreement with data. However, this is not surprising as it is simply reflecting the SU(3) symmetry of the coupling constants. In the FKR model the relative values of decay width for a given decay type are in less good agreement with data but this is due to the symmetry breaking introduced by using physical masses throughout. This problem is particularly acute in the case of

Table 2. Amplitudes for decays by emission of a pseudoscalar meson.

Decay type	Quark amplitude $-\mathrm{i}\langle \mathrm{f}\vert\partial_\mu A^q_\mu\vert\mathrm{i}\rangle$	Antiquark amplitude $-\mathrm{i}\langle \mathrm{f}\vert\partial_\mu \bar{A}^q_\mu\vert\mathrm{i}\rangle$
$1^{--} \to 0^{-+}0^{-+}$	$-\frac{m_1 m_2}{m_0} F\Lambda_2 q\,.\,\epsilon_1$	$\frac{m_1 m_2}{m_0} F\Lambda_2 q\,.\,\epsilon_1$
$2^{+-} \to 0^{-+}0^{-+}$	$\frac{m_1 m_2}{m_0} F\Lambda_2 q_\mu q_\nu T^{2\mu\nu}$	$\frac{m_1 m_2}{m_0} F\Lambda_2 q_\mu q_\nu T^{2\mu\nu}$
$2^{+-} \to 1^{--}0^{-+}$	$\frac{-\mathrm{i}F}{m_1 m_2\sqrt{\Omega}}(\Sigma+\Omega)T_2^{\alpha\beta}\mathscr{E}_{\alpha\mu\nu\lambda}(P_1^\mu P_2^\nu \epsilon_2^{\lambda *})q_\beta$	$\frac{\mathrm{i}F}{m_1 m_2\sqrt{\Omega}}(\Sigma+\Omega)T_2^{\alpha\beta}\mathscr{E}_{\alpha\mu\nu\lambda}(P_1^\mu P_2^\nu \epsilon_2^{\lambda *})q_\beta$
$1^{+-} \to 1^{--}0^{-+}$	$\frac{-F\sqrt{\Omega}}{m_0}\left[\epsilon_1\,.\,\epsilon_2^*\left(\frac{p_1\,.\,p_2}{m_1 m_2}\Delta_2\right) + q\,.\,\epsilon_1 q\,.\,\epsilon_2^*\left(\frac{m_1 m_2}{\Omega}\Lambda_1+\frac{\Delta_2}{m_1 m_2}\right)\right]$	$\frac{-F\sqrt{\Omega}}{m_0}\left[-\epsilon_1\,.\,\epsilon_2^*\left(\frac{p_1\,.\,p_2}{m_1 m_2}\Delta_2\right) + q\,.\,\epsilon_1 q\,.\,\epsilon_2^*\left(\frac{m_1 m_2}{\Omega}\Lambda_1-\frac{\Delta_2}{m_1 m_2}\right)\right]$
$1^{++} \to 1^{--}q^{-+}$	$\frac{F}{\sqrt{(2\Omega)}m_1^2 m_2}\{\epsilon_1\,.\,\epsilon_2^*[(\Omega-\Sigma)Y-2\Delta_1 p_1\,.\,p_2\Omega] + q\,.\,\epsilon_1 q\,.\,\epsilon_2^*[(p_1\,.\,p_2-2m_1^2)\Omega + p_1\,.\,p_2\Sigma]\}$	$\frac{F}{\sqrt{(2\Omega)}m_1^2 m_2}\{\epsilon_1\,.\,\epsilon_2^*[-(\Omega+2m_2^2)Y + 2\Delta_1 p_1\,.\,p_2\Omega]+q\,.\,\epsilon_1 q\,.\,\epsilon_2^*[-(p_1\,.\,p_2 - 2m_1^2)\Omega+2m_1^2 m_2^2]\}$

Abbreviations:

$$\Lambda_i = \left[1+\left(\frac{m_0}{m_i}\right)^2\right] \qquad \Delta_i = m_i^2 - p_1\,.\,p_2 \qquad Y = [m_1^2 m_2^2 - (p_1\,.\,p_2)^2]$$

$$\Sigma = m_1^2+m_2^2+m_3^2 \qquad p_1\,.\,p_2 = \tfrac{1}{2}(m_1^2+m_2^2-m_3^2)$$

$T^{2\alpha\beta}$ is a spin-two wavefunction.

decays into two pseudoscalar mesons where different results are obtained depending on which meson is replaced by the axial current. The use of unbroken masses avoids this asymmetry.

The value of the coupling constant f is expected to be close to that given by PCAC theory, that is $f_T = f_{\pi NN}^{1/2}/m_\pi g_A = 1{\cdot}65$. Using this value for the coupling constant the decay widths are found to be too large and reduction to a value of $f = 1{\cdot}46\ \mathrm{GeV}^{-1}$ is indicated in order to give a good fit to the data. We display the decay width data, our best fit, and that of FKR (Γ_F) in table 3.

The results for the vector and tensor decay widths are all within 20% of the experimental values, which is reasonably good considering that we have not introduced symmetry breaking, and it is an improvement on FKR. For the $2^+ \to 0^-1^-$ decays our results are less good than FKR's, however we could still modify our results by generating the axial vector current from a conserved current as in equation (5.11).

In the other two types of decay considered, $1^{+-} \to 1^-0^-$ and $1^{++} \to 1^-0^-$ we again have the problem that the underlying current is not conserved and consequently that these results could be improved. The results as calculated however are at least the correct order of magnitude and are no worse than those of FKR. On the other hand the

Table 3. Widths for decays by emission of a pseudoscalar meson.

Decay type	State	Mode	Γ(MeV)	Γ^{exp}(MeV)	Γ^{FKR}(MeV)
$1^- \to 0^-0^-$	φ(1019)	$K\bar{K}$	3·14	2·5±0·3	9
		ρπ	0	<0·6	0
	ω(784)	ππ	0	0·13±0·03	0
	K*(892)	Kπ	46·2	50·1±1·1	59·5
		πK	46·2	50·1±1·1	144
	ρ(765)	ππ	117	146±10	142
$1^{+-} \to 1^-0^-$	B(1235)	ωπ	66	120±20	76·5
$1^{++} \to 1^-0^-$	K*(1240)	K*π	47	~100	54
	A_1(1070)	ρπ	100	200–400	145
$2^+ \to 0^-0^-$	f′(1514)	$K\bar{K}$	62	40±10	93
		ππ	0	~0	0
	f(1260)	$K\bar{K}$	5·1	8±5	12
		ππ	153	130±12	220
	K*(1420)	Kπ	54·7	55±6	78
		πK	54·7	55±6	126
		Kη	2	~2	4·5
		ηK	2	~2	3·6
	A_2(1300)	ηπ	13·8	15±1·5	20
		πη	13·8	15±1·5	40
		$K\bar{K}$	7·4	4·7±1	15
$2^+ \to 0^-1^-$	f′(1514)	$\bar{K}K^* + K\bar{K}^*$	9·3	<14	13·5
	K*(1420)	K*π	18·5	29·5±6	20
		ρK	5·7	9·2±3	7
		ωK	1·4	4·4±2	1·8
	A_2(1300)	ρπ	53	72±7	60

helicity properties, which depend critically on the type of coupling are worse in our model as can be seen from table 4.

The unfavourable comparison with experiment for the K**(1240) → K*π and $A_1 \to \rho\pi$ decays is perhaps mitigated by possible contamination of the resonances by the Deck effect (Particle Data Group 1973).

Table 4. Predicted ratio of helicity amplitudes for $1^+ \to 1^-0^-$ decays compared with data from Colglazier and Rosner (1972).

Decay	T_{00}/T_{11}	$(T_{00}/T_{11})^{exp}$	$(T_{00}/T_{11})^{FKR}$
B → ωπ	1·0	0·2 → 0·7	0·19
A_1 → ρπ	1·0	2·0 → 1·1	1·3

6. Conclusions

We do not test the harmonic oscillator character of our wavefunctions because only the lowest two states are used, it is the spin structure we test. However, our basic assumption of harmonic oscillator forces gave rise to the spin structure so that any success of the model is attributable to this assumption.

The large components of the wavefunctions are analagous to boosted non-relativistic wavefunctions and occur in all relativistic models. The main difference arises from whether m_{quark} or m_{hadron} is regarded as the fundamental mass. The novel feature of our model is the inclusion of a spin–orbit term in the small components which, nonetheless, did not cause splitting of the trajectories. This extra orbital term is needed to produce some of our good results, the least ambiguous case being the electromagnetic decays where the orbital term contributes half of the result. We regard this success in the electromagnetic case as our most important result, as it does not depend on arbitrary parameters.

The significance of the results for the decay widths is harder to evaluate because the prescription which replaces the pseudoscalar interaction by the divergence of the axial vector current is not quite as well established as the corresponding formalism for electromagnetic decays. Further, we do not adhere strictly to PCAC theory as we use a coupling constant different (but not by much) from the theory.

Given that the relative success for any one decay type just depends on the SU(3) symmetry and the prescription to use unbroken masses, the fact that the coupling constant is an independent parameter means that table 3 contains only four independent results. Also, because the data for the $1^{++} \rightarrow 1^{-}0^{-}$ decay is ambiguous, there are really only three numbers to compare with experiment. Of these, two compare well while the third, $B \rightarrow \pi\omega$ is a factor of two out. This lack of success may be due to our decision not to modify the axial vector current so that it corresponds to a conserved electromagnetic current.

To summarize, we have proposed an equation in which the internal quark motion is closely related to that in a non-relativistic model but is treated in a covariant way. Our model is most closely related to that of FKR but has the advantage of incorporating spin in a more dynamical way. In compensation for their simple treatment of spin, FKR are able to include baryons in their scheme. The extension of our model to include baryons is clearly the next step.

Acknowledgments

The support of the SRC is acknowledged. We thank the referee for enlightening comments.

Appendix

A brief outline of the method of solution to equation (2.13)

We consider equation (2.13) in the rest frame excluding the time-like excitations. Defining $\mu = m_0/\surd(2\omega)$ and $\epsilon = m/\surd(2\omega)$ the quark–quark equation is

$$[\mu - \tfrac{1}{2}\epsilon(\gamma_0^{(1)} + \gamma_0^{(2)})]|\psi\rangle = (\boldsymbol{\alpha}_+ \,.\, \boldsymbol{a} + \boldsymbol{\alpha}_- \,.\, \boldsymbol{a}^\dagger)|\psi\rangle. \tag{A.1}$$

Writing the wavefunction

$$|\psi\rangle = \begin{pmatrix} |1\rangle & |2\rangle \\ |3\rangle & |4\rangle \end{pmatrix}$$

where $|i\rangle, i = 1, 2, 3, 4$, are 2×2 Pauli wavefunctions, equation (A.1) becomes the set of coupled equations:

$$\begin{aligned}
(\mu-\epsilon)|1\rangle &= \tfrac{1}{2}\boldsymbol{a}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|3-2\rangle+\tfrac{1}{2}\boldsymbol{a}^\dagger\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|3+2\rangle \\
(\mu+\epsilon)|4\rangle &= \tfrac{1}{2}\boldsymbol{a}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|3-2\rangle-\tfrac{1}{2}\boldsymbol{a}^\dagger\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|3+2\rangle \\
\mu|2\rangle &= \tfrac{1}{2}\boldsymbol{a}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|4-1\rangle+\tfrac{1}{2}\boldsymbol{a}^\dagger\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|4+1\rangle \\
\mu|3\rangle &= \tfrac{1}{2}\boldsymbol{a}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)|4-1\rangle-\tfrac{1}{2}\boldsymbol{a}^\dagger\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|4+1\rangle
\end{aligned} \tag{A.2}$$

where $|3+2\rangle = |3\rangle+|2\rangle$ etc. With some manipulation equations (A.2) may be reduced to a single eigenvalue equation

$$(\epsilon^2-\mu^2)|4+1\rangle = (\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{a}^\dagger(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{a}|4+1\rangle+(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{a}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{a}^\dagger|4+1\rangle. \tag{A.3}$$

PC = −1 solutions

Parity and charge conjugation (equation (3.3)) imply that

$$|4+1\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}|\phi\rangle.$$

It may now be verified that $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{a}^\dagger|4+1\rangle = 0$ and hence equation (A.3) reduces to the eigenvalue equation

$$(\epsilon^2-\mu^2-4\boldsymbol{a}^\dagger\cdot\boldsymbol{a})|\phi\rangle = 0. \tag{A.4}$$

From angular momentum conservation $|\phi\rangle = |N:J, J_z\rangle$ with the notation of § 4.1. The full solution for the rest frame is now easily constructed. Equation (4.1) is obtained by boosting into a general frame.

PC = +1 solutions

Parity and charge conjugation imply in this case that $|4+1\rangle$ is a pure spin one object and hence we may write

$$|4+1\rangle = \begin{pmatrix} |u\rangle & |s\rangle \\ |s\rangle & |d\rangle \end{pmatrix}.$$

Now equation (A.3) reduces to three degenerate eigenvalue equations

$$(\epsilon^2-\mu^2-4(\boldsymbol{a}^\dagger\cdot\boldsymbol{a}+2))|f\rangle = 0 \qquad f = u, d, s. \tag{A.5}$$

Clearly, this demands that $|f\rangle$ be an eigenstate of the number operator, but for $j = l\pm 1$ both orbital angular momenta l and $l\pm 2$ are allowed in general and relative proportion is not determined. This is the ambiguity referred to in the text. In the solution (4.5) we assumed that the rest frame solution contained no contamination from spin zero, which is equivalent to asserting that $|3+2\rangle = 0$. We may now substitute back into equations (A.2) and obtain the full solution for the rest frame. Equation (4.5) is obtained by boosting into a general frame.

We also note that equation (A.5) contains no spin–orbit interaction and so there is no spin–orbit splitting. However, the small components $|2\rangle$ and $|3\rangle$ will contain spin–orbit coupling terms and this is simply a consequence of parity conservation.

References

Bohm M, Joos H and Krammer M 1973 *Nucl. Phys.* B **51** 397–433
Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)
Colglazier E W and Rosner J L 1972 *Nucl. Phys.* B **27** 349–71
Choudhury D K and Mitra A N 1970 *Phys. Rev.* D **1** 351–7
Copley L A, Karl G and Obryk E 1969 *Phys. Lett.* **29B** 117–20
Dalitz R H 1966 *Proc. 13th Int. Conf. on High Energy Physics, Berkeley* (Berkeley: University of California Press)
——1967 *Lectures at 2nd Hawaii Topical Conf. on Particle Physics* (Honolulu: University of Hawaii Press)
Ebel G, Mullinsiefen A, Pilkuhn H, Steiner F, Wegener M, Michael C, Peterson J, Roos M, Martin B R, Oades G and de Swart J J 1971 *Nucl. Phys.* B **33** 317–78
Faiman D and Hendry A W 1968 *Phys. Rev.* **173** 1720–9
——1969 *Phys. Rev.* **180** 1572–7
Feynman R P, Kislinger M and Ravndal F 1971 *Phys. Rev.* D **3** 2706–32
Gell-Mann M 1964 *Phys. Lett.* **8** 214–5
Greenberg D W 1964 *Phys. Rev. Lett.* **13** 598–602
Gudeus T 1969 *Phys. Rev.* **184** 1788–92
Katyal D L and Mitra A N 1970 *Phys. Rev.* D **1** 338–50
Le Yaouanc A, Oliver L, Pene P and Raynal J-C 1972 *Orsay Preprint* LPTHE 72/6
Llewellyn-Smith C H 1969 *Ann. Phys., NY* **53** 521–58
Melosh H J 1973 *Thesis* Caltech
Particle Data Group 1973 *Rev. Mod. Phys.* **45** S1–175
Ramond P 1971 *Phys. Rev.* D **3** 2415–8
Rosner J L 1972 *Phys. Rev.* D **6** 1781–96
Sundaresan M K and Watson P J S 1970 *Ann. Phys., NY* **59** 375–407
Susskind L 1968 *Phys. Rev. Lett.* **21** 944–5
Van Royen and Weisskopf V 1967a *Nuovo Cim.* **50** 617–45
——1967b *Nuovo Cim.* **51** 583